Int. J. Solids Structures Vol. 30, No. 7, pp. 981–1000, 1993
Printed in Great Britain

0020–7683/93 \$6.00 + .00

AN IMPROVED PLATE THEORY OF {1, 2}-ORDER FOR THICK COMPOSITE LAMINATES

ALEXANDER TESSLER
NASA Langley Research Center, Computational Mechanics Branch, Mail Stop 240, Hampton, VA 23665-5225, U.S.A.

(*Received* 2 *September* 1991; *in revised form* 6 *October* 1992)

Abstract—A new two-dimensional laminate plate theory is developed for the linear elastostatic analysis of thick composite plates. The theory employs *equivalent single-layer* assumptions for the displacements, transverse shear strains, and transverse normal stress. The inplane and transverse displacements are respectively linear and quadratic expansions through the laminate thickness, where the low-order expansion coefficients correspond to the variables of Reissner's first-order shear-deformable theory. The transverse shear strains and transverse normal stress are assumed to be quadratic and cubic respectively through the thickness; they are expressed in terms of the kinematic variables of the theory by means of a least-squares compatibility requirement for the transverse strains and explicit enforcement of exact traction boundary conditions on the top and bottom plate surfaces. Application of the virtual work principle results in the 10th-order equations of equilibrium and associated Poisson boundary conditions. A major advantage of this theory over other higher-order theories lies in its perfect suitability for finite element approximation: simple C^0 continuous interpolations for the kinematic variables of the first-order theory (and, optionally, C^{-1} interpolations for the two higher-order displacements) are needed to formulate effective and robust two-dimensional plate elements capable of full three-dimensional ply-by-ply strain and stress recovery within the framework of general-purpose finite element codes. In assessing the predictive capability of the theory, analytic solutions for the problem of cylindrical bending are derived and compared with exact three-dimensional elasticity results and those of the earlier version of the {1, 2}-order plate theory.

NOMENCLATURE

$A_{ij}, \bar{A}_{ij}$	membrane plate rigidities
$B_{ij}, \bar{B}_{ij}$	membrane-bending coupling plate rigidities
$C_{ij}^{(k)}$	elastic stiffness coefficients for kth ply
$D_{ij}, \bar{D}_{ij}$	bending plate rigidities
E_L, E_T	longitudinal and transverse Young's moduli
G_{LT}, G_{TT}	shear moduli
G_{ij}	transverse shear plate rigidities
C^p	the class of continuous functions possessing p-order continuous derivatives at element interfaces; $p = -1$ implies discontinuous functions at element interfaces
C_u, C_σ	intersections of the cylindrical edge surface with the midplane where displacements and traction resultants are prescribed, respectively
$2h$	plate thickness
L	half-wavelength of transverse pressure loading
L_{ij}	linear differential operator
N	number of plies in laminate
N_{ij}	membrane and inplane shear force resultants
M_{ij}	bending and twisting moment resultants
N_z, M_z	transverse force and moment resultants
Q_x, Q_y	transverse shear force resultants
q^+, q^-	applied transverse loads
S^+, S^-	top and bottom plate surfaces
S_m	reference surface of plate
S_σ	part of the cylindrical edge surface where tractions are prescribed
$\bar{T}_i (i = x, y, z)$	prescribed edge tractions
u, v	midplane displacements in x and y directions
$u_i (i = x, y, z)$	Cartesian displacement components
w, w_1, w_2	components of the transverse displacement, u_z
δ	variational operator
$\varepsilon_{ij0}, \gamma_{ij0}, \kappa_{ij0}$	reference surface plate strains and curvatures
$\theta_i (i = x, y)$	bending cross-sectional rotations
$\xi \in [-1, 1]$	dimensionless thickness coordinate
$\sigma_{ij}^{(k)}, \tau_{ij}^{(k)}, \varepsilon_{zz}^{(k)}$	stress and strain components in the kth ply
$\varepsilon_{ij}, \gamma_{ij}, \sigma_{zz}$	average strains and stress in plate theory
ν_{LT}, ν_{TT}	Poisson ratios
$(\,)_{,q}, (\partial/\partial q)$	partial differentiation with respect to Cartesian coordinates $q = x, y, z$.

INTRODUCTION

Thick-section organic-matrix composites offer superior structural characteristics in the design of primary load-carrying components in advanced aircraft and combat ground vehicles. This is due to their superior strength- and stiffness-to-weight ratios as compared to homogeneous metallic structures. One of the difficulties in designing with thick-section composite laminates is the ability to adequately model their structural response and to predict failure under service loads. The effects of relatively large thickness, as compared to the wavelength of loading, as well as reduced stiffness and strength in the transverse shear and transverse normal material directions, can contribute to significant deformations and matrix dominated failure modes such as delamination and transverse cracking.

Most general-purpose finite element codes incorporate shear-deformable plate and shell theories of $\{1, 0\}$-order (i.e. first-order theories) in which the inplane and transverse displacement components vary through the laminate thickness as linear and constant functions, respectively. The underlying mathematical basis for such elements, which enables the development of simple and effective formulations, is the requirement of C^0 continuity† for the five engineering displacement variables describing stretching, bending and shear deformations. From the viewpoint of the finite element method, theories requiring higher than C^0 continuity (e.g. the Poisson–Kirchhoff theory with the C^1-continuous transverse displacement), are significantly less attractive [e.g. see Hughes (1987)].

Reissner (1944, 1945) derived his first-order shear-deformable theory for homogeneous isotropic plates in equilibrium using an assumed-stress approach; Mindlin (1951) proposed an analogous displacement-based theory for elastodynamics which also includes the rotary inertia effect. These theories take transverse shear deformations into account in some weighted-average sense and require the fulfilment of physically meaningful Poisson-type boundary conditions along the cylindrical plate edges—the prescription of the basic displacement and rotation variables or their associated force and moment resultants.

In modeling the mechanics of laminated composites with elasticity and plate/shell theories, the most widely and almost exclusively used level of abstraction is known as *effective-modulus* or *ply-elasticity* theory. In this level of approximation, the laminate is regarded as an assembly of homogeneous, anisotropic plies, perfectly bonded together at their interfaces, leading to a piecewise constant representation of the stiffness properties within each ply. [Refer to Pagano and Soni (1989) for the discussion on the utility of effective-modulus theory for composites modeling.]

Employing the effective-modulus abstraction, Yang *et al.* (1966) extended Reissner's theory to laminated plates. The theory incorporates individual ply properties into a smeared *equivalent single-layer* plate stiffness. Many variants of the first-order theory of this type have since been proposed; refer to the reviews of extensive literature by Reissner (1985), Librescu and Reddy (1986), Noor and Burton (1989) and Reddy (1990a). Numerous higher-order shear-deformable theories that assume the inextensibility in the transverse direction while accounting for the transverse normal stress have also been developed [e.g. refer to Librescu *et al.* (1987) and references therein]. This class of theories employs cubic assumptions for the inplane displacement components and a constant distribution for the transverse displacement.

The inclusion of both strain and stress in the transverse normal direction—which can be particularly pronounced in moderate-to-thick laminates and which can markedly contribute to interlaminar failure in the form of delamination—necessarily involves a higher-order expansion across the thickness of the transverse displacement component. The simplest displacement theory of this type is of order $\{1, 2\}$, using linear inplane and quadratic transverse displacement approximations over the laminate thickness. Hildebrand *et al.* (1949) and Naghdi (1957), using respectively displacement and mixed variational formulations, derived their $\{1, 2\}$-order theories for homogeneous elastic shells; Whitney and Sun (1974) extended Hildebrand *et al.*'s formulation to laminated composite shells. Lo *et al.* (1977) developed a $\{3, 2\}$-order displacement theory (cubic inplane and quadratic

† The notation C^p refers to the class of admissible, piecewise continuous functions possessing p-order continuous derivatives for extremizing functionals containing derivatives of order $p+1$.

transverse displacements) for homogeneous and laminated plates; Reddy (1990b) established the correlation between several versions of the $\{3, 2\}$-order theory. From the perspective of the finite element method, the class of higher-order theories suffers from a large number of C^0-continuous kinematic variables [and, in some instances, C^1-continuous variables, Reddy (1990b)] and the presence of higher-order edge-boundary conditions. The ramification is that a notably larger number of degrees of freedom and a special attention to the boundary conditions are required in order to construct meaningful, properly posed finite element models based on such theories. Accordingly, higher-order theories have not found their way into the general-purpose finite element codes.

Several *layer-wise* theories have been proposed [e.g. refer to Reddy (1989), Babuska *et al.* (1992), and references therein] which produce potentially accurate interlaminar strain and stress predictions at the expense of computationally intensive analyses. In *layer-wise* theories, the number of independent field variables is directly proportional to the number of plies in a laminate; hence, the analysis is computationally prohibitive to model realistic structures and thus limited to only relatively small (local) domains of interest. Similar computational considerations restrict the modeling of thick laminates with three-dimensional elements. In practical computations, finite elements based on *layer-wise* theories or three-dimensional elements are employed in a global-local fashion only in the local regions of particular significance, with the global analysis domains modeled by first-order shear-deformable plate and shell elements.

Recently, Tessler (1991a) derived a $\{1, 2\}$-order, orthotropic plate theory which has certain analytic advantages and is devoid of the computational drawbacks of other theories of this order of approximation. While being sufficiently accurate in the range of thin to thick plates, the theory retains the simplicity and computational advantages of the first-order theory. The ensuing developments included analytic and finite element analyses of elastic beams, Tessler (1991b), the extension to a $\{1, 2\}$-order orthotropic shell theory, Tsui and Tessler (1991), the extension to an *equivalent single-layer* theory for laminated composite plates in elastostatics, Tessler and Saether (1991), and elastodynamics, Tessler *et al.* (1992).

In these theories, the inplane displacement components vary linearly over the thickness as in the first-order shear-deformable theory. The transverse displacement is a quadratic function of a special form, containing three displacement variables that are independent of the thickness coordinate; one of these variables is Reissner's (first-order theory) weighted-average transverse deflection. Also, independent expansions of the transverse shear and normal strains are introduced which allow explicit enforcement of zero shear-traction boundary conditions on the top and bottom plate faces. [By comparison, in the $\{1, 2\}$-order theories of Hildebrand *et al.* (1949), Naghdi (1957), and Whitney and Sun (1974), these traction-free boundary conditions are not satisfied.] To derive a displacement theory, the transverse strains are made to be least-squares compatible (over the thickness) with those derived from strain–displacement relations of elasticity theory, thus expressing the strains and stresses in terms of the displacement variables. Applying the principle of virtual work, the analytic theory for bending and stretching of plates (shells) results in the 10th-order partial differential equations of equilibrium accompanied by the physical Poisson boundary conditions, as in the first-order theory. The variational principle also yields two additional boundary conditions which are identically satisfied provided that the transverse normal edge traction is a certain parabolic function of the thickness coordinate.

For the finite element approximation, the theory requires only C^0-continuous functions for the basic variables of the first-order theory and, optionally, C^{-1}-continuous interpolations for the two higher-order displacement variables. Hence, the theory lends itself perfectly to the development of simple, robust, and computationally efficient formulations similar to the first-order elements [e.g. Tessler and Hughes (1983, 1985)], with the advantage of full three-dimensional deformations and strain and stress recovery.

In the paper by Tessler and Saether (1991), the plate theory was satisfactorily assessed on the problem of cylindrical bending of a symmetric *angle-ply* carbon/epoxy laminate ($[30/-30]_s$), where comparisons were made to the exact elasticity solution given by Pagano (1970). In the subsequent analyses of *cross-ply* laminates, both symmetric and unsymmetric

([0/90]$_s$ and [0/90]), it was found that for *thin* laminates the transverse normal stress, obtained from a three-dimensional Hooke's law, erroneously exhibits large discontinuities at the ply interfaces. This error can be linked to the transverse normal strain, which is assumed in the theory to vary continuously across the thickness. It is noted that any error in the transverse normal stress is not particularly evident in angle-ply laminates.

To improve the accuracy in the transverse normal stress, a new version of the $\{1,2\}$-order theory is formulated for the linear elastostatic analysis of laminated composite plates. The formulation departs from that in Tessler and Saether (1991) only in the treatment of the transverse normal strain and stress. Herein, the transverse normal stress is expanded as a cubic polynomial through the laminate thickness, in contrast to a similar approximation for the transverse normal strain in the previous effort. This assumption can be justified by examining available exact elasticity solutions which, at the ply interfaces, exhibit continuity of both the stress and its gradient (see below). The transverse shear strains are taken to be parabolic, ensuring the satisfaction of zero shear tractions on the top and bottom bounding surfaces. The three transverse strain components are enforced to be least-squares compatible (through the laminate thickness) with the corresponding strains derived from the elasticity strain-displacement relations. The resulting variational principle, the differential equations of equilibrium, the associated boundary conditions, and the finite element continuity requirements remain unchanged from the previous theory.

In assessing the accuracy of the proposed theory, a well-established three-dimensional elasticity solution for a cylindrically bent laminate of Pagano (1970) is employed. Results for several different laminations and span-to-thickness ratios demonstrate that the inaccuracy in the transverse normal stress, which was present in the previous version of the theory, is completely resolved in this new $\{1,2\}$-order theory.

{1,2}-ORDER PLATE THEORY

Displacement expansions

Consider a laminated composite plate of uniform thickness $2h$ composed of N orthotropic plies whose transverse elastic constitutive properties do not differ appreciably. The displacement vector $\mathbf{u} = (u_x, u_y, u_z)$ corresponding to the Cartesian coordinates (x, y, z) can be approximated in the thickness direction z with functions that vary continuously over the laminate thickness. In accordance with Tessler and Saether (1991), the $\{1,2\}$-order displacement approximations which account for both transverse shear and transverse normal deformations employ linear expansions for the inplane components u_x and u_y and a special quadratic form for the transverse displacement u_z:

$$\begin{aligned} u_x(x,y,z) &= u(x,y)+h\xi\theta_y(x,y),\\ u_y(x,y,z) &= v(x,y)+h\xi\theta_x(x,y),\\ u_z(x,y,z) &= w(x,y)+\xi w_1(x,y)+(\xi^2-\tfrac{1}{5})w_2(x,y), \end{aligned} \tag{1}$$

where $\xi = z/h \in [-1, 1]$ is the dimensionless thickness coordinate and $\xi = 0$ identifies the reference midplane position. Of the seven kinematic variables in (1), $u(x, y)$, $v(x, y)$, $w(x, y)$, $\theta_x(x, y)$ and $\theta_y(x, y)$ are the conventional Reissner–Mindlin plate variables defined as the weighted-average quantities:

$$w = \frac{3}{4h}\int_{-h}^{h} u_z(1-\xi^2)\,\mathrm{d}z, \quad (u,v) = \frac{1}{2h}\int_{-h}^{h}(u_x,u_y)\,\mathrm{d}z, \quad (\theta_x,\theta_y) = \frac{3}{2h^3}\int_{-h}^{h}(u_y,u_x)z\,\mathrm{d}z, \tag{2}$$

where $u(x, y)$ and $v(x, y)$ are the midplane displacements along the x and y axes, $\theta_x(x, y)$ and $\theta_y(x, y)$ are the rotations of the normal about the x and y axes. Note the special form of u_z in (1): the specific choice of the thickness distribution $(\xi^2-\frac{1}{5})$ associated with the

$w_2(x, y)$ term allows $w(x, y)$ to be the weighted-average displacement of Reissner's first-order theory† [as defined in (2)], with $w_1(x, y)$ and $w_2(x, y)$ admitting a parabolic thickness distribution of the transverse displacement, u_z. These higher-order displacements can also be interpreted as the normalized strain and curvature in the thickness direction

$$w_1/h = u_{z,z}(z=0) = \varepsilon_{z0}, \quad w_2/h^2 = \tfrac{1}{2}u_{z,zz} = \kappa_{z0}. \tag{3}$$

Note that (1) can reasonably be regarded as some weighted-average approximations of the exact elasticity displacements which, for laminated composites, are C^0-piecewise (ply-level) continuous, possessing discontinuous thickness gradients at the ply interfaces.

Three-dimensional Hooke's law

The three-dimensional, anisotropic Hooke's law for the kth ply, whose inplane principal material directions are not, in general, coincident with the laminate axes, is expressed in the mixed form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \varepsilon_{zz} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} & \hat{C}_{16} & R_{13} & 0 & 0 \\ \hat{C}_{12} & \hat{C}_{22} & \hat{C}_{26} & R_{23} & 0 & 0 \\ \hat{C}_{16} & \hat{C}_{26} & \hat{C}_{66} & R_{63} & 0 & 0 \\ -R_{13} & -R_{23} & -R_{63} & S_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{45} \\ 0 & 0 & 0 & 0 & C_{45} & C_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \sigma_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \tag{4}$$

with

$$\hat{C}_{ij}^{(k)} = C_{ij}^{(k)} - R_{i3}^{(k)} C_{j3}^{(k)}, \quad R_{i3}^{(k)} = C_{i3}^{(k)} S_{33}^{(k)}, \quad S_{33}^{(k)} = 1/C_{33}^{(k)} \quad (i = 1, 2, 6),$$

where the 13 three-dimensional elastic stiffness coefficients, $C_{ij}^{(k)}$, corresponding to the x–y laminate coordinates are related to the nine elastic constants with respect to the material symmetry axes by way of a tensor transformation (Lekhnitskii, 1963). The coefficients $\hat{C}_{ij}^{(k)}(i, j = 1, 2, 6)$ may be regarded as the moduli relative to the generalized plane-stress condition, i.e. when the transverse normal stress is ignored. The use of (4) is particularly useful in the present plate theory, separating the average strains and stress (ε_{xx}, ε_{yy}, γ_{xy}, σ_{zz}, γ_{yz}, γ_{xz}) from the ply-dependent quantities that are superscribed with the (k) index.

Inplane strains

The inplane strains are customarily obtained from the linear strain–displacement relations of elasticity theory using (1):

$$\varepsilon_{xx} = \varepsilon_{x0} + z\kappa_{x0}, \quad \varepsilon_{yy} = \varepsilon_{y0} + z\kappa_{y0}, \quad \gamma_{xy} = \gamma_{xy0} + z\kappa_{xy0}, \tag{5}$$

where the plate strain and curvature variables, which are independent of the thickness coordinate, are defined as

$$\begin{aligned} [\varepsilon_{x0}, \quad \varepsilon_{y0}, \quad \gamma_{xy0}] &= [u_{,x}, \quad v_{,y}, \quad u_{,y} + v_{,x}], \\ [\kappa_{x0}, \quad \kappa_{y0}, \quad \kappa_{xy0}] &= [\theta_{y,x}, \quad \theta_{x,y}, \quad \theta_{x,x} + \theta_{y,y}]. \end{aligned} \tag{6}$$

By virtue of the displacement approximations in (1), the inplane strains (6) should be regarded as some average representations of the true strains. These average strains are inherently C^1-continuous across the laminate thickness (having a point-wise continuous

† The specific choice of the kinematic variables, combined with the assumptions concerning transverse stresses and strains, (7)–(10), leads to a 10th-order system of differential equations of equilibrium, associated Poisson boundary conditions, and the displacement variational principle that is perfectly suited for the finite element method.

function and its first gradient); by contrast, in laminated composites, the exact elasticity inplane strains have the same C^0 continuity characteristics as the inplane displacements.

Transverse strains and stresses

The present theory departs from that in Tessler and Saether (1991) only in the manner in which the transverse normal strain, ε_{zz}, and stress, σ_{zz}, are developed. Here, σ_{zz}, γ_{xz} and γ_{yz}, which are regarded as average quantities in the same sense as the inplane strains in (5), are expanded independently over the laminate thickness as†

$$\sigma_{zz} = \sum_{n=0}^{3} \sigma_{zn}\xi^n, \quad \gamma_{iz} = \sum_{n=0}^{2} \gamma_{in}\xi^n \quad (i = x, y), \tag{7}$$

where $\sigma_{zn} = \sigma_{zn}(x, y)$ and $\gamma_{in} = \gamma_{in}(x, y)$ are, as yet, undetermined.

To determine the expansion coefficients in (7), the procedure in Tessler and Saether (1991) is followed: The stress field is required to satisfy traction-free boundary conditions on the top and bottom plate surfaces, S^+ and S^-:

$$\tau_{xz}^{(k)}(x, y, \pm h) = \tau_{yz}^{(k)}(x, y, \pm h) = 0. \tag{8a}$$

From the third transverse equilibrium equation of elasticity theory,† using (8a), there result two homogeneous constraint equations on σ_{zz}, i.e.

$$\sigma_{zz,z}(x, y, \pm h) = 0. \tag{8b}$$

Equations (8) determine two coefficients in each expansion in (7). The remaining unknown coefficients are obtained from the following variational conditions: The transverse strains are enforced to be least-squares compatible with those computed directly from the strain–displacement relations of elasticity theory using the assumed displacements (1) (*direct strains*), i.e.

$$\text{minimize} \int_{-h}^{h} [\varepsilon_{zz}^{(k)} - u_{z,z}]^2 \, dz,$$

$$\text{minimize} \int_{-h}^{h} [\gamma_{iz} - (u_{z,i} + u_{i,z})]^2 \, dz \quad (i = x, y), \tag{9}$$

where the *direct strains* have the form:

$$u_{z,z} = \varepsilon_{z0} + 2z\kappa_{z0}, \quad u_{z,i} + u_{i,z} = \gamma_{i0} + z\varepsilon_{z0,i} + \left(z^2 - \frac{h^2}{5}\right)\kappa_{z0,i}, \tag{9a}$$

$$[\gamma_{x0}, \quad \gamma_{y0}] = [w_{,x} + \theta_y, \quad w_{,y} + \theta_x]. \tag{9b}$$

The minimization in (9) is performed with respect to the σ_{zn} and γ_{in} expansion coefficients, (7); $\varepsilon_{zz}^{(k)}$ is obtained from (4) using (5) and σ_{zz} defined in (7). The resulting transverse normal stress and transverse shear strains are

$$\sigma_{zz} = \sigma_{z0} + (\xi - \xi^3/3)\sigma_{z1}, \quad \gamma_{iz} = \frac{5}{4}(1 - \xi^2)\gamma_{i0} \quad (i = x, y), \tag{10}$$

where the σ_{z0} and σ_{z1} coefficients are functions of the elastic stiffness coefficients, $C_{qp}^{(k)}$, and the plate strain components in (3) and (6). The explicit forms of σ_{z0} and σ_{z1} are summarized

† The polynomial thickness expansions (7) are field-consistent in the sense of the transverse equilibrium equation of three-dimensional elasticity, $\tau_{xz,x}^{(k)} + \tau_{yz,y}^{(k)} + \sigma_{zz,z} = 0$. For homogeneous plates, the present approach and that in Tessler and Saether (1991) are equivalent.

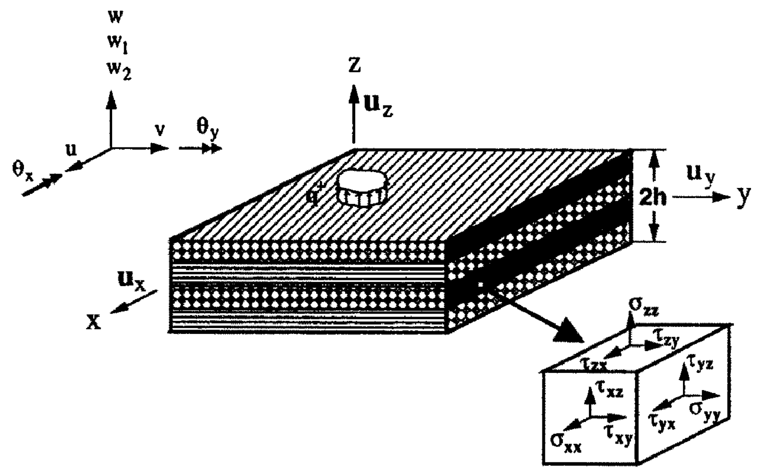

Fig. 1. Notation for $\{1, 2\}$-order plate theory.

in Appendix A. Observe that w_1 and w_2 are absent from the transverse shear strains in (10), which is a consequence of the special form of u_z in (1) and the fulfillment of (9).

With the average quantities for the inplane strains, the transverse normal stress, and the transverse shear strains completely determined in terms of the plate kinematic variables, (5) and (10), the three-dimensional Hooke's law, (4), can be applied to obtain the ply-dependent stress and strain components in any kth ply of the laminated plate.

Remark 1. The assumptions (7) can further be justified by examining exact stress distributions which are available for a cylindrical bending problem solved by Pagano (1970) (see Fig. 2). Figure 3 shows exact transverse shear and normal stress variations across the laminate thickness; these results correspond to [0], $[30/-30]_s$, $[0/90]_s$ and [0/90] carbon/epoxy laminates. Apparently, the transverse normal stress and its gradient through the thickness are continuous at the ply interfaces, and this is consistent for all lamination sequences examined. This naturally suggests that a cubic polynomial approximation for σ_{zz} should be an improvement over an analogous expansion for ε_{zz}, with the latter assumption resulting in a ply-discontinuous σ_{zz} (Tessler and Saether, 1991). (The results in the Results and Discussion section support this assertion.) On the other hand, the transverse shear stress solutions for the angle-ply and cross-ply laminates exhibit appreciable slope discontinuity at the ply interfaces (Fig. 3). Hence, approximating these latter stresses with polynomials that ensure continuous slope at the ply interfaces may not necessarily be advantageous, especially when such approximations are of a relatively low order (e.g. a parabola).

Remark 2. The average shear strains in (7) are generally not suitable for the recovery of transverse shear stresses from the constitutive law (4); this is because they produce discontinuities in the shear stresses at the ply interfaces where, theoretically, these stresses maintain continuity. [The exception is, of course, the homogeneous case (Tessler, 1991a).] Customarily, the issue of transverse shear stress recovery is resolved in a somewhat indirect and ad hoc manner: That is, these stresses are computed fairly accurately by integrating

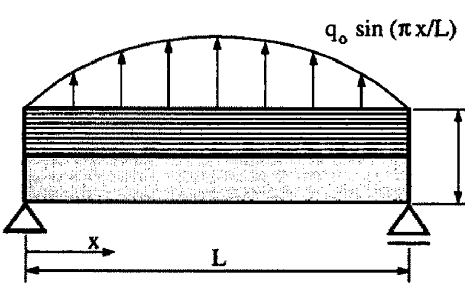

Fig. 2. Cylindrical bending of infinite laminated plate.

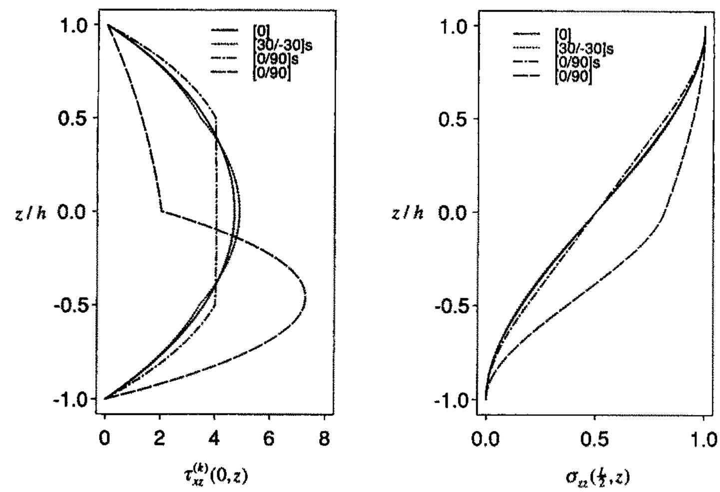

Fig. 3. Exact thickness distributions of $\tau_{xz}^{(k)}(0,z)$ and $\sigma_{zz}^{(k)}(L/2,z)$ in Gr/Ep laminates of $L/2h = 10$.

appropriate three-dimensional elasticity equations of equilibrium [e.g. see Tessler and Saether (1991)].

Remark 3. When the bounding plate surfaces S^+ and S^- are subject to the nonvanishing transverse shear, $\tau_{iz}^{\pm}$, and normal, $q^{\pm}$, tractions, (8) can be rewritten to fulfill these boundary conditions exactly, i.e. $\tau_{iz}^{(k)}(x, y, \pm h) = \tau_{iz}^{\pm}\,(i = x, y)$, and $\sigma_{zz}(x, y, \pm h) = q^{\pm}$. Note that the last of these equations can be used instead of (8b) in the present case as well.

Application of virtual work principle

The remainder of the formulation employs the three-dimensional statement of virtual work† which, neglecting body forces, may be written as

$$\int_V [\sigma_{xx}^{(k)}\delta\varepsilon_{xx} + \sigma_{yy}^{(k)}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz}^{(k)} + \tau_{xy}^{(k)}\delta\gamma_{xy} + \tau_{yz}^{(k)}\delta\gamma_{yz} + \tau_{xz}^{(k)}\delta\gamma_{xz}]\,\mathrm{d}V$$
$$-\int_{S^+} q^+(x,y)\delta u_z(x,y,h)\,\mathrm{d}x\,\mathrm{d}y + \int_{S^-} q^-(x,y)\delta u_z(x,y,-h)\,\mathrm{d}x\,\mathrm{d}y$$
$$-\int_{S_\sigma} [\bar{T}_x\delta u_x + \bar{T}_y\delta u_y + \bar{T}_z\delta u_z]\,\mathrm{d}s\,\mathrm{d}z = 0, \quad (11)$$

where $q^+(x, y)$ and $q^-(x, y)$ are the transverse normal pressure loads acting on S^+ and S^-, respectively, and defined positive in the positive directions of the transverse normal stress on the respective surfaces; $\bar{T}_i(i = x, y, z)$ denote the tractions prescribed on S_σ, which is a part of the cylindrical edge boundary.

Integrating (11) over the plate thickness results in the two-dimensional virtual work principle

$$\int_{S_m} [N_x\delta\varepsilon_{x0} + N_y\delta\varepsilon_{y0} + N_{xy}\delta\gamma_{xy0} + N_z\delta\varepsilon_{z0} + M_x\delta\kappa_{x0} + M_y\delta\kappa_{y0} + M_{xy}\delta\kappa_{xy0}$$
$$+ M_z\delta\kappa_{z0} + Q_x\delta\gamma_{x0} + Q_y\delta\gamma_{y0} - (q^+ - q^-)(\delta w + \tfrac{4}{5}\delta w_2) - (q^+ + q^-)\delta w_1]\,\mathrm{d}x\,\mathrm{d}y$$
$$-\oint_{C_\sigma} [\bar{N}_{xn}\delta u + \bar{N}_{yn}\delta v + \bar{M}_{xn}\delta\theta_y + \bar{M}_{yn}\delta\theta_x + \bar{Q}_{zn}\delta w + \bar{Q}_{z1}\delta w_1 + \bar{Q}_{z2}\delta w_2]\,\mathrm{d}s = 0, \quad (12)$$

† This statement may be regarded as a special *weak form* of the principle of virtual work when the least-squares compatibility requirement (9), which was used in deriving the transverse strains, is employed.

where C_σ is the intersection between the middle surface, S_m, and S_σ. The stress resultants are obtained by integrating appropriate stress components over the laminate thickness (Appendix B) yielding the plate constitutive relations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \\ N_{xy} \\ M_x \\ M_y \\ M_z \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{16} & B_{11} & B_{12} & B_{13} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{23} & A_{26} & B_{21} & B_{22} & B_{23} & B_{26} & 0 & 0 \\ A_{13} & A_{23} & A_{33} & A_{36} & B_{31} & B_{32} & B_{33} & B_{36} & 0 & 0 \\ A_{16} & A_{26} & A_{36} & A_{66} & B_{61} & B_{62} & B_{63} & B_{66} & 0 & 0 \\ B_{11} & B_{21} & B_{31} & B_{61} & D_{11} & D_{12} & D_{13} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{32} & B_{62} & D_{12} & D_{22} & D_{23} & D_{26} & 0 & 0 \\ B_{13} & B_{23} & B_{33} & B_{63} & D_{13} & D_{23} & D_{33} & D_{36} & 0 & 0 \\ B_{16} & B_{26} & B_{36} & B_{66} & D_{16} & D_{26} & D_{36} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & G_{54} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{54} & G_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \varepsilon_{z0} \\ \gamma_{xy0} \\ \kappa_{x0} \\ \kappa_{y0} \\ \kappa_{z0} \\ \kappa_{xy0} \\ \gamma_{x0} \\ \gamma_{y0} \end{Bmatrix}, \tag{13}$$

where A_{ij}, B_{ij}, D_{ij} and G_{ij} are the plate elastic stiffness coefficients (see Appendix C). Note that for an arbitrary lamination sequence the membrane-bending coupling matrix, $\mathbf{B} = [B_{ij}]$, is unsymmetric, although the full constitutive matrix in (13) is symmetric. A generalization of Castigliano's first theorem for this plate theory, which relates the gradients of the strain energy with respect to the generalized (plate) strains to the appropriate stress resultants, can be readily established and is presented in Appendix D.

In (12), the terms associated with the variations δw_1 and δw_2 are grouped noting that $\delta\varepsilon_{z0} = (1/h)\delta w_1$ and $\delta\kappa_{z0} = (1/h^2)\delta w_2$; since these variations are completely arbitrary, their corresponding multiplicative terms must vanish independently, resulting in

$$N_z = h(q^+ + q^-), \quad M_z = \frac{4h^2}{5}(q^+ - q^-) \quad \text{on} \quad S_m, \tag{14}$$

$$[\bar{Q}_{z1}, \quad \bar{Q}_{z2}] = \int_{-h}^{h} \bar{T}_z[\xi, \quad \xi^2 - \tfrac{1}{3}]\,\mathrm{d}z = [0,0] \quad \text{on} \quad C_\sigma. \tag{15}$$

Equations (14) represent the plate transverse normal equilibrium equations associated with the inclusion of σ_{zz} in the theory; eqns (15) are the natural boundary conditions which can be interpreted as the variationally consistent requirements on the thickness distribution of the prescribed traction, $\bar{T}_z$. It is readily seen that (15) are identically fulfilled provided $\bar{T}_z$ varies through the thickness as an even function of ξ, specifically as $(1-\xi^2)$.

Introducing (14) and (15) into (12) and (13) yields a variational principle having the basic form of the first-order theory, i.e.

$$\begin{aligned} \int_{S_m} [N_x\delta\varepsilon_{x0} + N_y\delta\varepsilon_{y0} + N_{xy}\delta\gamma_{xy0} + M_x\delta\kappa_{x0} + M_y\delta\kappa_{y0} \\ + M_{xy}\delta\kappa_{xy0} + Q_x\delta\gamma_{x0} + Q_y\delta\gamma_{y0} - (q^+ - q^-)\delta w]\,\mathrm{d}x\,\mathrm{d}y \\ - \oint_{C_\sigma} [\bar{N}_{xn}\delta u + \bar{N}_{yn}\delta v + \bar{M}_{xn}\delta\theta_y + \bar{M}_{yn}\delta\theta_x + \bar{Q}_{zn}\delta w]\,\mathrm{d}s = 0, \end{aligned} \tag{16}$$

as well as the reduced constitutive relations

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{16} & \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{16} & 0 & 0 \\ \bar{A}_{12} & \bar{A}_{22} & \bar{A}_{26} & \bar{B}_{21} & \bar{B}_{22} & \bar{B}_{26} & 0 & 0 \\ \bar{A}_{16} & \bar{A}_{26} & \bar{A}_{66} & \bar{B}_{61} & \bar{B}_{62} & \bar{B}_{66} & 0 & 0 \\ \bar{B}_{11} & \bar{B}_{21} & \bar{B}_{61} & \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{16} & 0 & 0 \\ \bar{B}_{12} & \bar{B}_{22} & \bar{B}_{62} & \bar{D}_{12} & \bar{D}_{22} & \bar{D}_{26} & 0 & 0 \\ \bar{B}_{16} & \bar{B}_{26} & \bar{B}_{66} & \bar{D}_{16} & \bar{D}_{26} & \bar{D}_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & G_{54} \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{54} & G_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \\ \kappa_{x0} \\ \kappa_{y0} \\ \kappa_{xy0} \\ \gamma_{x0} \\ \gamma_{y0} \end{Bmatrix} + \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \\ 0 \\ 0 \end{Bmatrix}, \tag{17}$$

where the expressions for the barred elastic stiffness coefficients and the N_i and $M_i (i = 1, 2, 6)$ terms are given in Appendix E. Note that N_i and M_i depend upon the transverse loads q^+ and q^- and laminate stiffness coefficients.

Integrating (16) by parts results in the remaining Euler–Lagrange plate equilibrium equations:

$$\begin{aligned} N_{x,x} + N_{xy,y} &= 0, \\ N_{xy,x} + N_{y,y} &= 0, \\ M_{x,x} + M_{xy,y} - Q_x &= 0, \\ M_{xy,x} + M_{y,y} - Q_y &= 0, \\ Q_{x,x} + Q_{y,y} + q^+ - q^- &= 0, \end{aligned} \tag{18}$$

and the Poisson boundary conditions:

(i) Kinematic (displacement) boundary conditions on C_u:

$$u = \bar{u}, \quad v = \bar{v}, \quad w = \bar{w}, \quad \theta_x = \bar{\theta}_x, \quad \theta_y = \bar{\theta}_y; \tag{19}$$

(ii) Natural (force) boundary conditions on C_σ:

$$\begin{aligned} & N_x n_x + N_{xy} n_y = \bar{N}_{xn}, \quad N_{xy} n_x + N_y n_y = \bar{N}_{yn}, \quad Q_x n_x + Q_y n_y = \bar{Q}_{zn}, \\ & M_x n_x + M_{xy} n_y = \bar{M}_{xn}, \quad M_{xy} n_x + M_y n_y = \bar{M}_{yn}, \\ & [n_x, n_y] = [\cos(x, n), \cos(y, n)], \end{aligned} \tag{20}$$

where n denotes the outward normal to C_σ, and C_u is the part of the cylindrical plate boundary where the displacements are prescribed.

Introducing (17) into (18) yields the equations of equilibrium in terms of the plate strains and curvatures

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} & L_{16} & 0 & 0 \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} & L_{26} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} & L_{36} & -G_{55} & -G_{54} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} & L_{46} & -G_{54} & -G_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{57} & L_{58} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \\ \kappa_{x0} \\ \kappa_{y0} \\ \kappa_{xy0} \\ \gamma_{x0} \\ \gamma_{y0} \end{Bmatrix} = - \begin{Bmatrix} N_{1,x} + N_{6,y} \\ N_{2,y} + N_{6,x} \\ M_{1,x} + M_{6,y} \\ M_{2,y} + M_{6,x} \\ q^+ - q^- \end{Bmatrix}, \tag{21}$$

where the load-dependent functions N_i and $M_i (i = 1, 2, 6)$ are given in Appendix E, and the linear differential operators L_{ij} are summarized in Appendix F.

Since the strain and curvature variables are linear differential operators of displacements, (21) constitutes a 10th-order, coupled membrane-bending theory. For homogeneous plates, the equilibrium equations (21) and the corresponding Poisson boundary conditions, (19) and (20), are coincident with the corresponding equations of Reissner's first-order theory (Tessler, 1991a). The compatibility equations for this theory are also coincident with those of the first-order theory (Appendix G).

REMARKS ON FEM APPROXIMATIONS

For the finite element approximations, either (12) or (16) can be employed as a variational framework. In (12), the basic variables of the first-order theory (u, v, w, θ_x and θ_y) have first-order derivatives and thus need only be approximated over the element domain with C^0-continuous trial functions. A unique feature of (12) is that w_1 and w_2 do not have derivatives, thus only requiring C^{-1}-continuous element-level approximations. This is in contrast to other previously mentioned $\{1, 2\}$ and even higher-order theories which necessitate at least C^0 continuity for all kinematic variables.

Equation (16) is a reduced variational form involving only C^0-continuous variables of the first-order theory. The two variational frameworks, (12) and (16), are nearly equivalent: applying (16) implies the exact fulfillment of the transverse normal equilibrium, (14), while the variational formulation according to (12) enforces (14) in the average variational sense. Note that the degrees-of-freedom associated with the w_1 and w_2 approximations, if (12) is employed, can be condensed out statically at the element level.

It is clear that the present theory offers the same computational advantages as its predecessor theory. In Tessler (1991) and Tessler and Saether (1991), a three-node plate element was developed using uniform assumptions for w_1 and w_2 and static condensation of these variables at the element level; further, the u, v, w, θ_x and θ_y variables employed the C^0-continuous, anisoparametric (variable-order) shape functions developed for Mindlin elements, by Tessler and Hughes (1985) and Tessler (1985). Other nodal-pattern elements can also be formulated using this methodology, for example, a four-node quadrilateral, as in Tessler and Hughes (1983).

RESULTS AND DISCUSSION

The present theory is evaluated on the problem of cylindrical bending of an infinite carbon/epoxy laminate subjected to a sinusoidal transverse pressure $q^+ = q_0 \sin(\pi x/L)$. The equations of equilibrium, (21), are solved exactly using $\sin(\pi x/L)$ and $\cos(\pi x/L)$

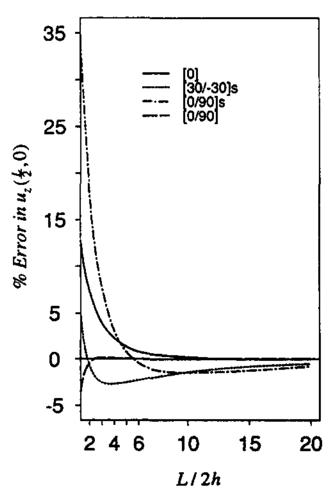

Fig. 4. Percentage error in maximum midplane deflection vs $L/2h$ ratio for various Gr/Ep laminates.

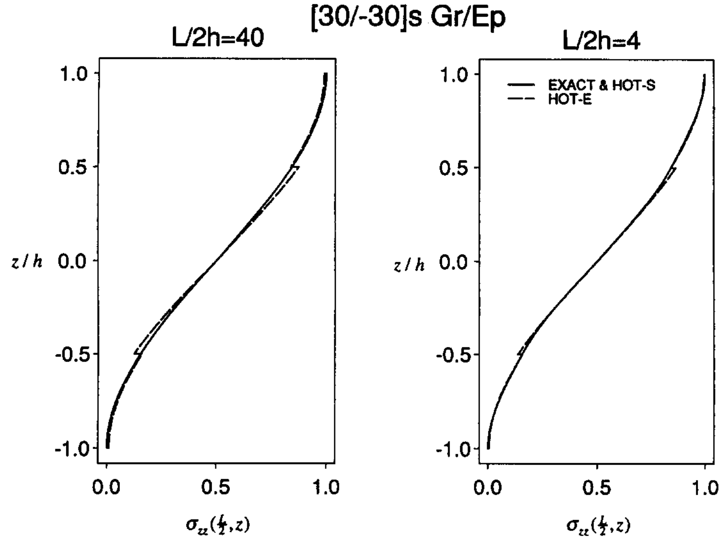

Fig. 5(a). Thickness distribution of $\sigma_{zz}(L/2, z)$ in $[30/-30]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

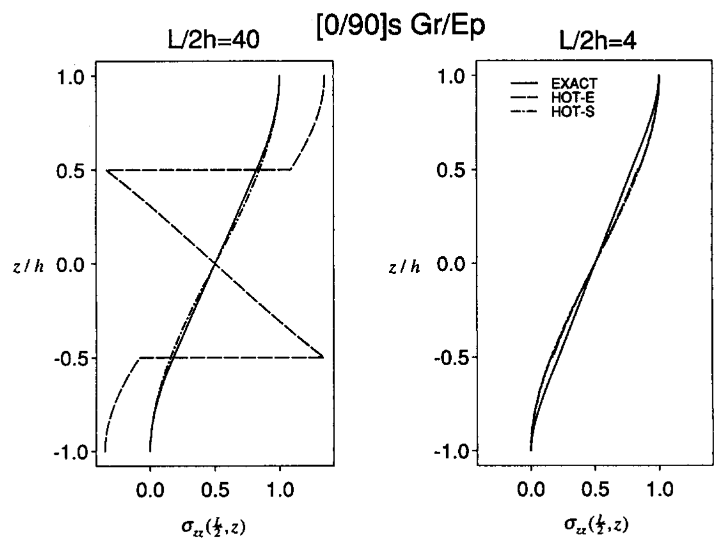

Fig. 5(b). Thickness distribution of $\sigma_{zz}(L/2, z)$ in $[0/90]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

expansions for the displacement variables, as in Tessler (1991a). The exact elasticity solution for this problem was first determined by Pagano (1970).

The ply material properties corresponding to a typical carbon/epoxy material are taken as

$$E_L = 25\times 10^6\,\text{psi},\quad E_T = 10^6\,\text{psi},\quad G_{LT} = 0.5\times 10^6\,\text{psi},$$
$$G_{TT} = 0.2\times 10^6\,\text{psi},\quad v_{LT} = v_{TT} = 0.25, \tag{22}$$

where L and T denote the longitudinal and transverse ply material directions, respectively.

As noted previously, this theory differs from that in Tessler and Saether (1991) only in the approximation for the transverse normal stress and strain. Naturally, the differences in the predictions by the two theories are expected to affect predominantly these quantities. This is confirmed by analytic solutions for the above problem yielding, for both theories, virtually identical displacements as well as the inplane and transverse shear stresses and strains.

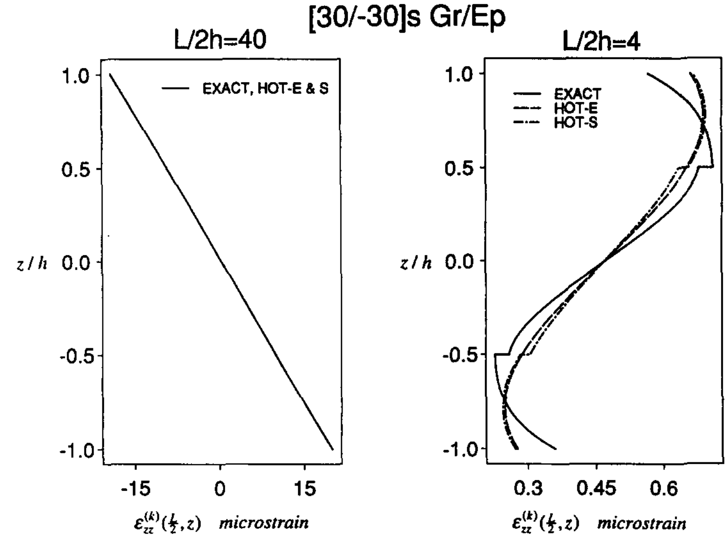

Fig. 6(a). Thickness distribution of $\varepsilon_{zz}^{(k)}(L/2, z)$ in $[30/-30]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

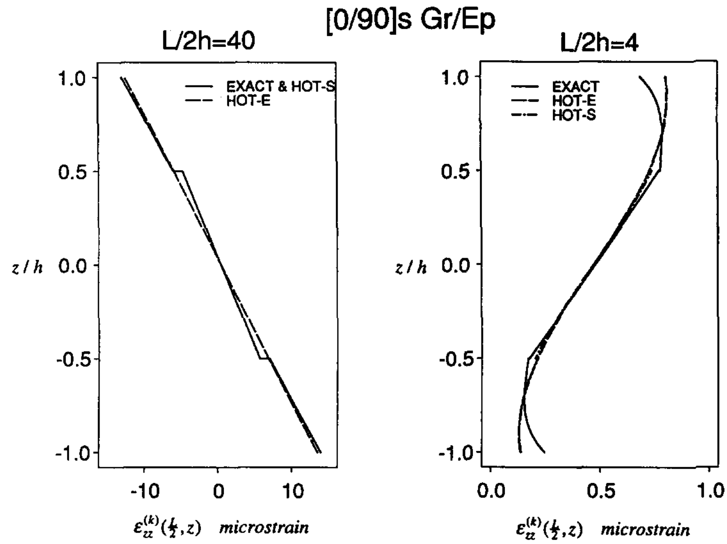

Fig. 6(b). Thickness distribution of $\varepsilon_{zz}^{(k)}(L/2, z)$ in $[0/90]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

First, as a measure of the overall laminate stiffness approximation, it is useful to examine the deflection results for various types of laminates; here, four laminated sequences are considered: an orthotropic [0] laminate, two symmetric laminates—a cross-ply $[0/90]_s$ laminate and an angle-ply $[30/-30]_s$ laminate—and an antisymmetric cross-ply laminate, [0/90]. Figure 4 depicts the percentage error in the maximum deflection, based on the comparison with the exact three-dimensional elasticity solution, versus the length-to-thickness ratio, $L/2h$. These results, which pertain to the present and Tessler and Saether (1991) theories, clearly show that as far as the deflection predictions are concerned, the engineering accuracy (i.e. error $\leqslant 5\%$) is attained for laminates with the ratio $L/2h \geqslant 4$.

The ensuing assessment is concerned with the quality of strain and stress approximations through the laminate thickness. Here, it suffices to focus on thin ($L/2h = 40$) and thick ($L/2h = 4$) symmetric laminates having cross-ply ($[0/90]_s$) and angle-ply ($[30/-30]_s$) laminate sequences. Note that laminates with the aspect ratio of $L/2h = 4$ were selected to challenge the limits of applicability of the theory in the thick regime.

Figures 5–8 compare stress and strain distributions obtained by the present theory (designated as HOT-S), the previous version of the $\{1,2\}$-theory [HOT-E, Tessler and

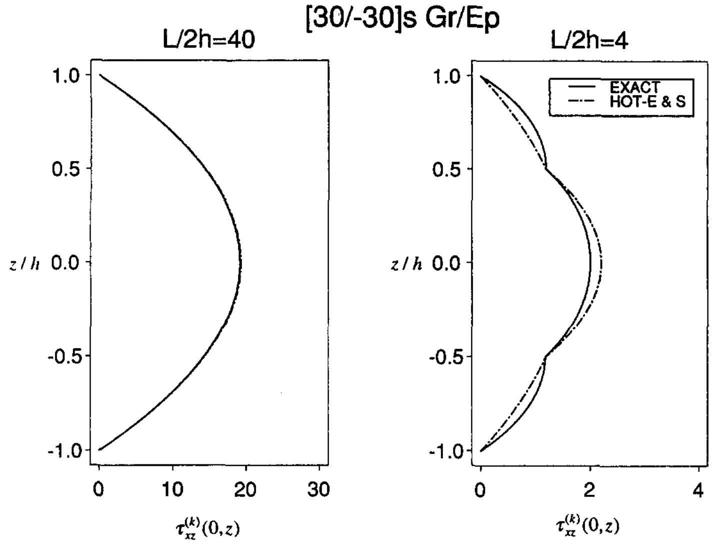

Fig. 7(a). Thickness distribution of $\tau_{xz}^{(k)}(0, z)$ in $[30/-30]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

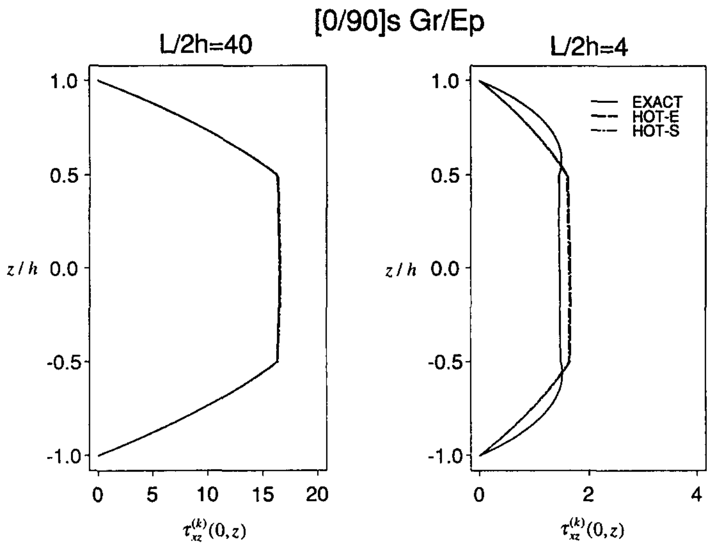

Fig. 7(b). Thickness distribution of $\tau_{xz}^{(k)}(0, z)$ in $[0/90]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

Saether (1991)], and Pagano's (1970) exact three-dimensional elasticity solutions (some of the exact results were obtained in this effort by applying Pagano's approach). Examining the σ_{zz} and ε_{zz} results in Figs 5 and 6, it is evident that in the angle-ply case, the HOT-E and HOT-S results yield nearly identical solutions of high accuracy, with HOT-E producing slightly discontinuous σ_{zz} at the $30°/-30°$ ply interface. Also, in the thick case, ε_{zz} is somewhat less accurate on the outer surfaces, although the general character of the distribution is consistent with the exact solution. In the *thin* cross-ply laminate, however, the HOT-E results are markedly in error, this compared to the HOT-S predictions which are excellent. This anomaly in HOT-E may be attributed to the continuous assumption for ε_{zz} [see eqns (12), Tessler and Saether (1991)], which contradicts the exact ε_{zz} exhibiting a relatively large discontinuity at the ply interfaces as does the present HOT-S solution [Fig. 6(b)].

The transverse shear stress distributions, τ_{xz}, are compared in Fig. 7. The plate-theory stresses are computed by integrating appropriate inplane stress gradients in the three-dimensional equations of equilibrium (Tessler and Saether, 1991). As shown in the figures, the plate-theory predictions and exact elasticity solutions compare very closely for thin

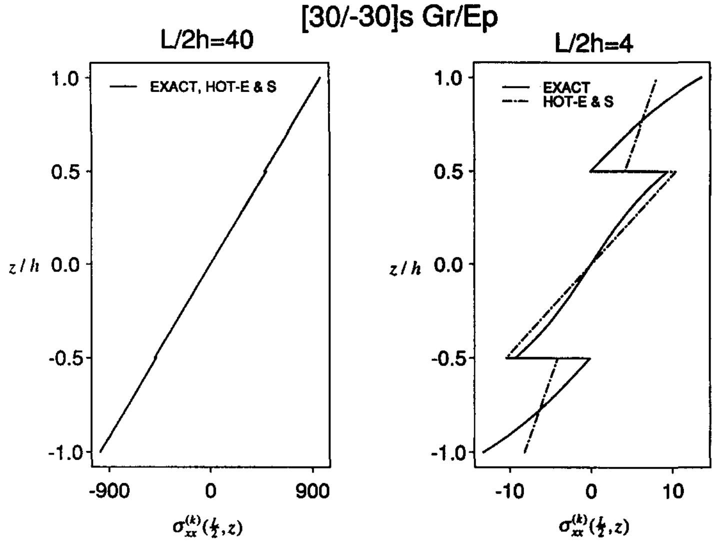

Fig. 8(a). Thickness distribution of $\sigma_{xx}^{(k)}(L/2, z)$ in $[30/-30]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

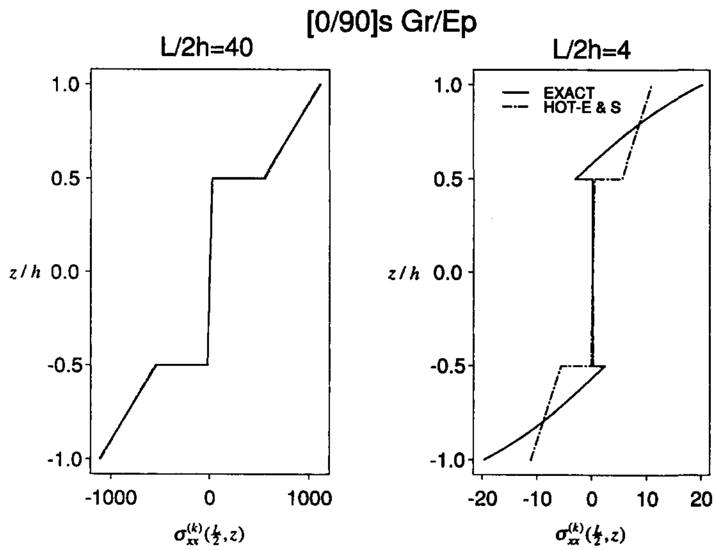

Fig. 8(b). Thickness distribution of $\sigma_{xx}^{(k)}(L/2, z)$ in $[0/90]_s$ Gr/Ep laminates; $L/2h = 40$ and 4.

laminates and reasonably well for thick laminates. Figure 8 depicts the σ_{xx} stress variation through the thickness. These results show that for thick laminates, the stress predictions in the outer plies are significantly underestimated. This is naturally due to the limiting *linear* assumption for the inplane displacements (1); in the thick regime, however, the exact inplane displacement and the associated inplane strain are appreciably nonlinear through the thickness (not shown).

CONCLUDING REMARKS

In this paper, a new version of a $\{1,2\}$-order laminate plate theory for the linear elastostatic analysis of thin and thick laminated composite plates has been presented which utilizes independent assumptions for the displacements, transverse shear strains and transverse normal stress.

The theory has both analytical and computational appeal. From the analytical standpoint, it is manifested by 10th-order differential equilibrium equations and associated engineering (Poisson-type) boundary conditions. For arbitrary laminations, the resulting plate equations of equilibrium couple the deformations due to inplane stretching, inplane shear, transverse normal stretching, and bending. The theory incorporates all components of strain and stress and is capable of full three-dimensional ply-by-ply recovery of these quantities. In the case of material homogeneity, the equilibrium equations reduce to those of Reissner's first-order theory. The theory offers quantitative improvements over the earlier version, particularly in recovering the transverse normal strain and stress. From the viewpoint of utilizing the theory within the finite element method, it offers the same advantages as the first-order theory: It is a variationally-based, displacement theory requiring C^0-continuous interpolations for the five kinematic variables of the first-order theory and, optionally, C^{-1} interpolations for the two higher-order displacements. For these reasons, the theory is ideally suited for implementation in any general-purpose finite element code.

In critically evaluating the predictive capability of the present theory by studying the problem of cylindrical bending of various laminated plates, it appears that the theory is well suited in the range of thin to moderately thick laminates. It is also apparent that, in order to best evaluate the modeling benefits and practical range of applicability of this theory, further assessments should be carried out modeling the response of actual multilayered composite structures, for example, a composite aircraft wing. For this reason, a suitable finite element formulation, such as in Tessler and Saether (1991), should be employed in order to conduct the necessary large-scale finite element analyses. In addition, highly refined (converged) benchmark solutions based on three-dimensional finite element models should also be developed for appropriate comparisons. Naturally, the ultimate evaluation of any mathematical abstraction such as the present theory rests on the correlation with experimental data.

Acknowledgements—The author wishes to thank Erik Saether of the Army Materials Technology Laboratory for his computational support in this effort.

REFERENCES

Babuska, I., Szabo, B. A. and Actis, R. L. (1992). Hierarchic models for laminated composites. *Int. J. Numer. Meth. Engng* **33**, 503–535.

Hildebrand, F. B., Reissner, E. and Thomas, G. B. (1949). Notes on the foundations of the theory of small displacements of orthotropic shells. NACA Technical Note No. 1833.

Hughes, T. J. R. (1987). *The Finite Element Method. Linear Static and Dynamic Finite Element Analysis.* Prentice-Hall, Englewood Cliffs, NJ.

Lekhnitskii, S. G. (1963). *Theory of Elasticity of an Anisotropic Elastic Body.* Holden Day, Oakland, CA.

Librescu, L., Khdeir, A. A. and Reddy, J. N. (1987). A comprehensive analysis of the state of stress of elastic anisotropic flat plates using refined theories. *Acta Mech.* **70**, 57–81.

Librescu, L. and Reddy, J. N. (1986). A critical review and generalization of transverse shear deformable anisotropic plate theories. *Euromech Colloquium 219. Refined Dynamical Theories of Beams, Plates and Shells and their Applications*, Kassel, Germany.

Lo, K. H., Christensen, R. M. and Wu, E. M. (1977). A higher-order theory of plate deformation: Part 1. Homogeneous plates; Part 2. Laminated plates. *J. Appl. Mech. ASME* **44**, 663–676.

Mindlin, R. D. (1951). Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *J. Appl. Mech. ASME* **18**, 31–38.

Naghdi, P. M. (1957). On the theory of thin elastic shells. *Q. Appl. Math.* **14**, 369–380.

Noor, A. K. and Burton, W. S. (1989). Assessment of shear deformable theories for multilayered composite plates. *Appl. Mech. Rev.* **42**, 1–12.

Pagano, N. J. (1970). Influence of shear coupling in cylindrical bending of anisotropic laminates. *J. Compos. Mater.* **4**, 330–343.

Pagano, N. J. and Soni, S. R. (1989). Models for studying free-edge effects. In *Interlaminar Response of Composite Materials* (Edited by N. J. Pagano), pp. 1–68. Elsevier, Amsterdam.

Reddy, J. N. (1989). On refined computational models of composite laminates. *Int. J. Numer. Meth. Engng* **27**, 361–382.

Reddy, J. N. (1990a). A review of refined theories of laminated composite plates. *Shock Vibr. Dig.* **22**(7), 3–17.

Reddy, J. N. (1990b). A general non-linear third-order theory of plates with moderate thickness. *Int. J. Non-Lin. Mech.* **25**(6), 677–686.

Reissner, E. (1944). On the theory of bending of elastic plates. *J. Math. Phys.* **23**, 184–191.

Reissner, E. (1945). The effect of transverse shear deformations on the bending of elastic plates. *J. Appl. Mech.* **12**, A69–A77.

Reissner, E. (1985). Reflections on the theory of elastic plates. *Appl. Mech. Rev.* **38**, 1453–1464.

Tessler, A. (1985). A priori identification of shear locking and stiffening in triangular Mindlin elements. *Comput. Meth. Appl. Mech. Engng* **53**, 183–200.

Tessler, A. (1991a). A higher-order plate theory with ideal finite element suitability *Comput. Meth. Appl. Mech. Engng* **85**, 183–205.
Tessler, A. (1991b). A two-node beam element including transverse shear and transverse normal deformations. *Int. J. Numer. Meth. Engng* **32**, 1027–1039.
Tessler, A. and Hughes, T. J. R. (1983). An improved treatment of transverse shear in the Mindlin-type four-node quadrilateral element. *Comput. Meth. Appl. Mech. Engng* **39**, 311–335.
Tessler, A. and Hughes, T. J. R. (1985). A three-node Mindlin plate element with improved transverse shear. *Comput. Meth. Appl. Mech. Engng* **50**, 71–101.
Tessler, A. and Saether, E. (1991). A computationally viable higher-order theory for laminated composite plates. *Int. J. Numer. Meth. Engng* **31**, 1069–1086.
Tessler, A., Saether, E. and Tsui, T. (1992). Vibration of thick composite laminates: Analytic theory and finite element approximation. *Proc. 33rd AIAA SDM Conference*, Dallas, TX.
Tsui, T. and Tessler, A. (1991). {1, 2}-order theory for homogeneous orthotropic shells. *Proc. Army Solid Mech. Conf.*, Plymouth, MA.
Whitney, J. M. and Sun, C. T. (1974). A refined theory for laminated anisotropic, cylindrical shells. *J. Appl. Mech. ASME* **41**, 471–476.
Yang, P. C., Norris, C. H. and Stavsky, Y. (1966). Elastic wave propagation in heterogeneous plates. *Int. J. Solids Structures* **2**, 665–684.

APPENDIX A: σ_{z0} AND σ_{z1} EXPANSION COEFFICIENTS

The expansion coefficients σ_{z0} and σ_{z1} for the transverse normal stress component (10) are computed according to the matrix equation:

$$\begin{Bmatrix}\sigma_{z0}\\ \sigma_{z1}\end{Bmatrix} = [\mathbf{q} \quad \hat{\mathbf{q}}]\begin{Bmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{\kappa}\end{Bmatrix}, \tag{A1}$$

where

$$\boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_{x0}, \quad \varepsilon_{y0}, \quad \varepsilon_{z0}, \quad \gamma_{xy0}], \quad \boldsymbol{\kappa}^{\mathrm{T}} = [\kappa_{x0}, \quad \kappa_{y0}, \quad \kappa_{z0}, \quad \kappa_{xy0}], \tag{A2}$$

and

$$\mathbf{q} = [q_{ij}] = \mathbf{s}^{-1}\mathbf{t}, \quad \hat{\mathbf{q}} = [\hat{q}_{ij}] = \mathbf{s}^{-1}\hat{\mathbf{t}} \quad (i = 1,2; j = 1,2,3,6). \tag{A3}$$

The matrices $\mathbf{s}$, $\mathbf{t}$ and $\hat{\mathbf{t}}$ are obtained as follows:

$$\mathbf{s} = [s_{ij}] \quad (i, j = 1,2),$$
$$[s_{11}, \quad s_{12} = s_{21}, \quad s_{22}] = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} [1, \quad \phi, \quad \phi^2] S_{33}^{(k)^2}\,\mathrm{d}z, \quad \phi = \xi - \tfrac{1}{3}\xi^3. \tag{A4}$$

$$\mathbf{t} = [t_{ij}] \quad (i = 1,2; j = 1,2,3,6),$$
$$t_{1j} = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} R_{j3}^{(k)} S_{33}^{(k)}\,\mathrm{d}z, \quad t_{2j} = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} \phi R_{j3}^{(k)} S_{33}^{(k)}\,\mathrm{d}z \quad (j = 1,2,6),$$
$$t_{13} = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} S_{33}^{(k)}\,\mathrm{d}z, \quad t_{23} = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} \phi S_{33}^{(k)}\,\mathrm{d}z. \tag{A5}$$

$$\hat{\mathbf{t}} = [\hat{t}_{ij}] \quad (i = 1,2; j = 1,2,3,6),$$
$$\hat{t}_{ij} = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} z R_{j3}^{(k)} S_{33}^{(k)}\,\mathrm{d}z, \quad \hat{t}_{2j} = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} z\phi R_{j3}^{(k)} S_{33}^{(k)}\,\mathrm{d}z \quad (j = 1,2,6),$$
$$\hat{t}_{13} = 2\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} z S_{33}^{(k)}\,\mathrm{d}z, \quad \hat{t}_{23} = 2\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} z\phi S_{33}^{(k)}\,\mathrm{d}z. \tag{A6}$$

APPENDIX B: PLATE STRESS RESULTANTS

The components of the stress resultants in the plate constitutive relations (13) are computed as follows:

Inplane and transverse normal forces

$$N^{\mathrm{T}} = [N_x, \quad N_y, \quad N_z, \quad N_{xy}]$$
$$= \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} [\sigma_{xx}^{(k)} + p_{13}^{(k)}\sigma_{zz}, \quad \sigma_{yy}^{(k)} + p_{23}^{(k)}\sigma_{zz}, \quad p_{33}^{(k)}\sigma_{zz}, \quad \tau_{xy}^{(k)} + p_{63}^{(k)}\sigma_{zz}]\,\mathrm{d}z,$$

where

$$p_{i3}^{(k)} = (q_{1i} + \phi q_{2i}) S_{33}^{(k)} - R_{i3}^{(k)}, \quad p_{33}^{(k)} = (q_{13} + \phi q_{23}) S_{33}^{(k)} \quad (i = 1,2,6). \tag{B1}$$

Bending moments

$$M^{\mathrm{T}} = [M_x, \quad M_y, \quad M_z, \quad M_{xy}]$$

$$= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\sigma_{xx}^{(k)} + \hat{p}_{13}^{(k)}\sigma_{zz}, \quad \sigma_{yy}^{(k)} + \hat{p}_{23}^{(k)}\sigma_{zz}, \quad \hat{p}_{33}^{(k)}\sigma_{zz}, \quad \tau_{xy}^{(k)} + \hat{p}_{63}^{(k)}\sigma_{zz}] z\,dz,$$

where

$$\hat{p}_{i3}^{(k)} = (\hat{q}_{1i} + \phi\hat{q}_{2i})S_{33}^{(k)} - R_{i3}^{(k)}, \quad \hat{p}_{33}^{(k)} = (\hat{q}_{13} + \phi\hat{q}_{23})S_{33}^{(k)} \quad (i = 1,2,6). \tag{B2}$$

Transverse shear forces

$$\mathbf{Q}^{\mathrm{T}} = [Q_x, \quad Q_y] = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{5}{4}(1-\xi^2)[\tau_{xz}^{(k)}, \quad \tau_{yz}^{(k)}]\,dz. \tag{B3}$$

Prescribed stress resultants along plate edges

$$[\bar{N}_{xn}, \quad \bar{N}_{yn}, \quad \bar{Q}_{zn}] = \int_{-h}^{h} [\bar{T}_x, \quad \bar{T}_y, \quad \bar{T}_z]\,dz, \quad [\bar{M}_{xn}, \quad \bar{M}_{yn}] = \int_{-h}^{h} [\bar{T}_x, \quad \bar{T}_y] z\,dz. \tag{B4}$$

Remarks. In (B1)–(B4), the constitutive coefficients $S_{33}^{(k)}$ and $R_{i3}^{(k)}$ are given in (4), q_{ij} and $\hat{q}_{ij}$ are computed in (A3)–(A6), and ϕ is given in (A4).

APPENDIX C: ELASTIC STIFFNESS COEFFICIENTS

The elastic stiffness coefficients appearing in the plate constitutive relations (13) are computed according to the relations:

A_{ij} *stiffness coefficients*

$$A_{ij} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\hat{C}_{ij}^{(k)} + (q_{1i} + \phi q_{2i})(q_{1j} + \phi q_{2j})S_{33}^{(k)}]\,dz \quad (i,j = 1,2,6),$$

$$A_{i3} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (q_{1i} + \phi q_{2i})(q_{13} + \phi q_{23})S_{33}^{(k)}\,dz \quad (i = 1,2,3,6). \tag{C1}$$

B_{ij} *stiffness coefficients*

$$B_{ij} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [z\hat{C}_{ij}^{(k)} + (q_{1i} + \phi q_{2i})(\hat{q}_{1j} + \phi\hat{q}_{2j})S_{33}^{(k)}]\,dz,$$

$$B_{33} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (q_{13} + \phi q_{23})(\hat{q}_{13} + \phi\hat{q}_{23})S_{33}^{(k)}\,dz,$$

$$B_{i3} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (q_{1i} + \phi q_{2i})(\hat{q}_{13} + \phi\hat{q}_{23})S_{33}^{(k)}\,dz,$$

$$B_{3i} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (q_{13} + \phi q_{23})(\hat{q}_{1i} + \phi\hat{q}_{2i})S_{33}^{(k)}\,dz \quad (i = 1,2,6). \tag{C2}$$

D_{ij} *stiffness coefficients*

$$D_{ij} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [z^2\hat{C}_{ij}^{(k)} + (\hat{q}_{1i} + \phi\hat{q}_{2i})(\hat{q}_{1j} + \phi\hat{q}_{2j})S_{33}^{(k)}]\,dz \quad (i,j = 1,2,6),$$

$$D_{i3} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\hat{q}_{1i} + \phi\hat{q}_{2i})(\hat{q}_{13} + \phi\hat{q}_{23})S_{33}^{(k)}\,dz, \quad D_{ij} = D_{ji} \quad (i = 1,2,3,6). \tag{C3}$$

G_{ij} *stiffness coefficients*

$$G_{ij} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\frac{5}{4}(1-\xi^2)\right]^2 C_{ij}^{(k)}\,dz \quad (i,j = 4,5). \tag{C4}$$

Remarks. In (C1)–(C4), the constitutive coefficients $S_{33}^{(k)}$, $C_{ij}^{(k)}$ and $\hat{C}_{ij}^{(k)}$ are given in (4). In a symmetrically laminated (balanced) plate, the B_{ij} coefficients vanish.

APPENDIX D: GENERALIZATION OF CASTIGLIANO'S FIRST THEOREM

The strain energy per unit of reference surface area is the integral over the thickness of the plate:

$$U = \tfrac{1}{2}\int_{-h}^{h} [\sigma_{xx}^{(k)}\varepsilon_{xx} + \sigma_{yy}^{(k)}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz}^{(k)} + \tau_{xy}^{(k)}\gamma_{xy} + \tau_{yz}^{(k)}\gamma_{yz} + \tau_{xz}^{(k)}\gamma_{xz}]\,dz, \tag{D1}$$

which upon integration takes the form:

$$U = \tfrac{1}{2}[N_x\varepsilon_{x0} + N_y\varepsilon_{y0} + N_z\varepsilon_{z0} + N_{xy}\gamma_{xy0} + M_x\kappa_{x0} + M_y\kappa_{y0} + M_z\kappa_{z0} + M_{xy}\kappa_{xy0} + Q_x\gamma_{x0} + Q_y\gamma_{y0}]. \tag{D2}$$

From (D2) and with the use of (13), these results a generalization of Castigliano's first theorem appropriate for this plate theory:

$$\begin{gathered}\left[\frac{\partial}{\partial\varepsilon_{x0}}, \frac{\partial}{\partial\varepsilon_{y0}}, \frac{\partial}{\partial\varepsilon_{z0}}, \frac{\partial}{\partial\gamma_{xy0}}\right](U) = [N_x, N_y, N_z, N_{xy}],\\ \left[\frac{\partial}{\partial\kappa_{x0}}, \frac{\partial}{\partial\kappa_{y0}}, \frac{\partial}{\partial\kappa_{z0}}, \frac{\partial}{\partial\kappa_{xy0}}\right](U) = [M_x, M_y, M_z, M_{xy}],\\ \left[\frac{\partial}{\partial\gamma_{x0}}, \frac{\partial}{\partial\gamma_{y0}}\right](U) = [Q_x, Q_y].\end{gathered} \tag{D3}$$

APPENDIX E: REDUCED CONSTITUTIVE EQUATIONS

The equations that follow are the elastic stiffness coefficients and the loading terms in the reduced constitutive relations, (17):

Components of $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ *and* $\bar{\mathbf{D}}$ *matrices*

- Diagonal Terms ($i = 1, 2, 6$)

$$\begin{aligned}\bar{A}_{ii} &= A_{ii} - (A_{i3}^2 D_{33} - 2A_{i3}B_{i3}B_{33} + A_{33}B_{i3}^2)/(A_{33}D_{33} - B_{33}^2),\\ \bar{B}_{ii} &= B_{ii} + [A_{i3}(D_{i3}B_{33} - B_{3i}D_{33}) + B_{i3}(B_{3i}B_{33} - D_{i3}A_{33})]/(A_{33}D_{33} - B_{33}^2),\\ \bar{D}_{ii} &= D_{ii} - (B_{3i}^2 D_{33} - 2B_{3i}D_{i3}B_{33} + A_{33}D_{i3}^2)/(A_{33}D_{33} - B_{33}^2).\end{aligned} \tag{E1}$$

- Off-Diagonal Terms ($i = 1, 2$; $j = 2, 6$; $i \neq j$)

$$\begin{aligned}\bar{A}_{ij} &= A_{ij} + [A_{i3}(B_{j3}B_{33} - A_{j3}D_{33}) + B_{i3}(A_{j3}B_{33} - B_{j3}A_{33})]/(A_{33}D_{33} - B_{33}^2),\\ \bar{B}_{ij} &= B_{ij} + [A_{i3}(D_{j3}B_{33} - B_{3j}D_{33}) + B_{i3}(B_{3j}B_{33} - D_{j3}A_{33})]/(A_{33}D_{33} - B_{33}^2),\\ \bar{B}_{ji} &= B_{ji} + [A_{j3}(D_{i3}B_{33} - B_{3i}D_{33}) + B_{j3}(B_{3i}B_{33} - D_{i3}A_{33})]/(A_{33}D_{33} - B_{33}^2),\\ \bar{D}_{ij} &= D_{ij} + [B_{3i}(D_{j3}B_{33} - B_{3j}D_{33}) + D_{i3}(B_{3j}B_{33} - D_{j3}A_{33})]/(A_{33}D_{33} - B_{33}^2).\end{aligned} \tag{E2}$$

Loading functions (i = 1, 2, 6)

$$\begin{aligned}N_i &= -h\left[(q^+ + q^-)(B_{i3}B_{33} - A_{i3}D_{33}) + \frac{4h}{5}(q^+ - q^-)(A_{i3}B_{33} - B_{i3}A_{33})\right]/(A_{33}D_{33} - B_{33}^2),\\ M_i &= -h\left[(q^+ + q^-)(D_{i3}B_{33} - B_{3i}D_{33}) + \frac{4h}{5}(q^+ - q^-)(B_{3i}B_{33} - D_{i3}A_{33})\right]/(A_{33}D_{33} - B_{33}^2).\end{aligned} \tag{E3}$$

Remarks. In (E1)–(E3), the plate constitutive coefficients A_{ij}, B_{ij} and D_{ij} are computed in Appendix C.

APPENDIX F: LINEAR DIFFERENTIAL OPERATORS L_{ij}

The following equations are the linear differential operators appearing in (21):

$$L_{11} = \bar{A}_{11}\frac{\partial}{\partial x} + \bar{A}_{16}\frac{\partial}{\partial y},\quad L_{12} = \bar{A}_{12}\frac{\partial}{\partial x} + \bar{A}_{26}\frac{\partial}{\partial y},\quad L_{13} = \bar{A}_{16}\frac{\partial}{\partial x} + \bar{A}_{66}\frac{\partial}{\partial y},$$

$$L_{14} = \bar{B}_{11}\frac{\partial}{\partial x} + \bar{B}_{61}\frac{\partial}{\partial y},\quad L_{15} = \bar{B}_{12}\frac{\partial}{\partial x} + \bar{B}_{62}\frac{\partial}{\partial y},\quad L_{16} = \bar{B}_{16}\frac{\partial}{\partial x} + \bar{B}_{66}\frac{\partial}{\partial y},$$

$$L_{21} = \bar{A}_{16}\frac{\partial}{\partial x} + \bar{A}_{12}\frac{\partial}{\partial y},\quad L_{22} = \bar{A}_{26}\frac{\partial}{\partial x} + \bar{A}_{22}\frac{\partial}{\partial y},\quad L_{23} = \bar{A}_{66}\frac{\partial}{\partial x} + \bar{A}_{26}\frac{\partial}{\partial y},$$

$$L_{24} = \bar{B}_{61}\frac{\partial}{\partial x} + \bar{B}_{21}\frac{\partial}{\partial y},\quad L_{25} = \bar{B}_{62}\frac{\partial}{\partial x} + \bar{B}_{22}\frac{\partial}{\partial y},\quad L_{26} = \bar{B}_{66}\frac{\partial}{\partial x} + \bar{B}_{26}\frac{\partial}{\partial y},$$

$$L_{31} = \bar{B}_{11}\frac{\partial}{\partial x}+\bar{B}_{16}\frac{\partial}{\partial y},\quad L_{32} = \bar{B}_{21}\frac{\partial}{\partial x}+\bar{B}_{26}\frac{\partial}{\partial y},\quad L_{33} = \bar{B}_{61}\frac{\partial}{\partial x}+\bar{B}_{66}\frac{\partial}{\partial y},$$

$$L_{34} = \bar{D}_{11}\frac{\partial}{\partial x}+\bar{D}_{16}\frac{\partial}{\partial y},\quad L_{35} = \bar{D}_{12}\frac{\partial}{\partial x}+\bar{D}_{26}\frac{\partial}{\partial y},\quad L_{36} = \bar{D}_{16}\frac{\partial}{\partial x}+\bar{D}_{66}\frac{\partial}{\partial y},$$

$$L_{41} = \bar{B}_{16}\frac{\partial}{\partial x}+\bar{B}_{12}\frac{\partial}{\partial y},\quad L_{42} = \bar{B}_{26}\frac{\partial}{\partial x}+\bar{B}_{22}\frac{\partial}{\partial y},\quad L_{43} = \bar{B}_{66}\frac{\partial}{\partial x}+\bar{B}_{62}\frac{\partial}{\partial y},$$

$$L_{44} = \bar{D}_{16}\frac{\partial}{\partial x}+\bar{D}_{12}\frac{\partial}{\partial y},\quad L_{45} = \bar{D}_{26}\frac{\partial}{\partial x}+\bar{D}_{22}\frac{\partial}{\partial y},\quad L_{46} = \bar{D}_{66}\frac{\partial}{\partial x}+\bar{D}_{26}\frac{\partial}{\partial y},$$

$$L_{57} = G_{55}\frac{\partial}{\partial x}+G_{54}\frac{\partial}{\partial y},\quad L_{58} = G_{54}\frac{\partial}{\partial x}+G_{44}\frac{\partial}{\partial y}, \tag{F1}$$

where the elastic stiffness coefficients $\bar{A}_{ij}$, $\bar{B}_{ij}$, $\bar{D}_{ij}$ and G_{ij} are given in Appendices C and E.

APPENDIX G: PLATE THEORY COMPATIBILITY EQUATIONS

The theory is based on 10 strain measures expressed in terms of seven kinematic variables. These strain–displacement relations are given in (3), (6) and (9b) and are summarized below:

$$\begin{aligned}
[\varepsilon_{x0},\quad \varepsilon_{y0},\quad \gamma_{xy0}] &= [u_{,x},\quad v_{,y},\quad u_{,y}+v_{,x}],\\
[\kappa_{x0},\quad \kappa_{y0},\quad \kappa_{xy0}] &= [\theta_{y,x},\quad \theta_{x,y},\quad \theta_{x,x}+\theta_{y,y}],\\
[\gamma_{x0},\quad \gamma_{y0}] &= [w_{,x}+\theta_{y},\quad w_{,y}+\theta_{x}],\\
[\varepsilon_{z0},\quad \kappa_{z0}] &= [w_1/h,\quad w_2/h^2].
\end{aligned} \tag{G1}$$

The transverse normal strain measures (ε_{z0} and κ_{z0}) are determined in terms of the *primary* strain measures (ε_{x0}, ε_{y0}, γ_{xy0}, κ_{x0}, κ_{y0} and κ_{xy0}) with the use of (13) and (14). The relations can be expressed as

$$\begin{Bmatrix}\varepsilon_{z0}\\ \kappa_{z0}\end{Bmatrix} = \frac{1}{A_{33}D_{33}-B_{33}^2}\begin{bmatrix} D_{33} & -B_{33}\\ -B_{33} & A_{33}\end{bmatrix}$$

$$\times\left[\begin{Bmatrix} h(q^+ + q^-)\\ (4h^2/5)(q^+ - q^-)\end{Bmatrix} - \begin{bmatrix} A_{13} & A_{23} & A_{36} & B_{31} & B_{32} & B_{36}\\ B_{13} & B_{23} & B_{36} & D_{13} & D_{23} & D_{36}\end{bmatrix}\begin{Bmatrix}\varepsilon_{x0}\\ \varepsilon_{y0}\\ \gamma_{xy0}\\ \kappa_{x0}\\ \kappa_{y0}\\ \kappa_{xy0}\end{Bmatrix}\right]. \tag{G2}$$

Since ε_{z0} and κ_{z0} and, consequently, w_1 and w_2, are explicitly dependent upon the *primary* strains, they can be classified as the *auxiliary* variables of the theory. Eliminating the primary strains from (G1) results in the plate compatibility equations:

$$\begin{aligned}
\kappa_{xy0,xy} &= \kappa_{y0,xx}+\kappa_{x0,yy},\\
2\kappa_{x0,y} &= \frac{\partial}{\partial x}(-\gamma_{y0,x}+\gamma_{x0,y}+\kappa_{xy0}),\\
2\kappa_{y0,x} &= \frac{\partial}{\partial y}(-\gamma_{x0,y}+\gamma_{y0,x}+\kappa_{xy0}),\\
\gamma_{xy0,xy} &= \varepsilon_{x0,yy}+\varepsilon_{y0,xx}.
\end{aligned} \tag{G3}$$

These necessary and sufficient conditions for single-valuedness of the plate displacement variables are identical to those of the first-order theory (Mindlin, 1951).